KANTON
LUZERN

Bildungs- und Kulturdepartement
Kantonsschule Alpenquai Luzern

Written Matura Exam 2024

Subject	Mathematics Basic Course
Teacher	Roman Oberholzer roman.oberholzer@sluz.ch
Class	G20k, G20l
Date of the exam	Friday, 24th of May, 2024
Time	180 minutes
Aids allowed	- "Mathematics Formulary", Adrian Wetzel - A dictionary (book, no electronic translator) - TI-30X Pro Multiview or MathPrint (no handbook)
Instructions	- Importance is attached to a proper and clear representation. - Write each exercise on a separate sheet of paper. - All solutions must show the steps leading to the result. - Put your personal number, your name and your class on every sheet of paper.
Maximum points per exercise	Exercise 1: 11 Exercise 2: 12.5 Exercise 3: 11.5 Exercise 4: 12 **Total: 47** 42 points are required for a grade of 6.
Number of pages	5 (including title page)

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Surname, First name Class Number

Exercise 1	a	b	c	d	Points
Calculus I	6	1	2	2	**11**

We consider the function $f(x) = \frac{2x^2 + 9x - 18}{x^2}$ with the first two derivatives

$$f'(x) = \frac{-9x + 36}{x^3} \qquad f''(x) = \frac{18x - 108}{x^4}$$

a. Determine the domain, the zeroes, the stationary points (maximum and minimum points), the limits at the not-defined x-values and for $x \to \pm\infty$, and the asymptotes of f and then draw the graph of the function f for $-14 \leq x \leq 15$. Sketch the asymptotes as well. *Units: 1 square per unit.*

b. Show that the function $F(x) = 9 \cdot \ln(x) + 2x + \frac{18}{x} + c$ is an antiderivative of f.

c. Now, we consider the more general antiderivative of $F(x) = 9 \cdot \ln(x) + ax + \frac{b}{x}$, where c = 0 and x > 0. Find the values of a and b in such a way that the antiderivative F has the slope m = 8 at the point P(1|3).

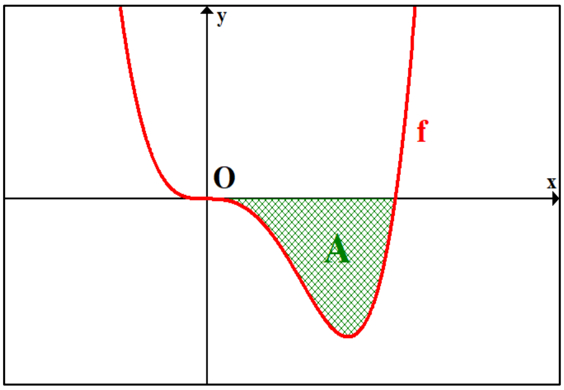
d. Determine the equation of the tangent from point O(0|0) at the graph of f in the first quadrant. *Round the answer to three digits*.

Exercise 2 Calculus II	a	b	c	d	Points
	2	1	5	4.5	12.5

We consider the function

$$f(x) = \frac{1}{10} \cdot \left(x^4 - 4x^3\right)$$

as sketched in the graph at the right.



a. What is the size of the acute intersection angle between the graph of f and the x-axis at the positive zero of the graph of f?

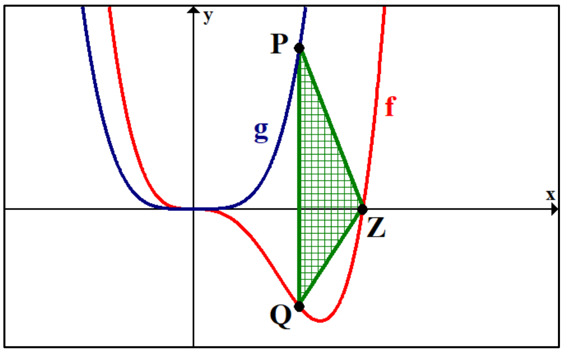
b. The shaded area A between the graph of the function f and the x-axis in the 4^{th} quadrant rotates about the x-axis. Determine the volume of the obtained solid, without finding and evaluating the antiderivative.

c. Now, we consider the tangent at the inflection point of the graph of f in the 4th quadrant. This tangent, together with the graph of f and the positive x-axis enclose a region. Calculate its area, without finding and evaluating the antiderivative.

d. In this exercise, we consider the function

$$g(x) = \frac{1}{10} \cdot x^4$$

besides the function f.



Between the graphs of f and g, the triangle ΔPQZ is inscribed, where the side PQ is parallel to the y-axis. Point P lies on the graph of g, point Q on the graph of f, and point Z is the zero of the graph of f on the positive x-axis.

Find the coordinates of point P in order for the area of the triangle ΔPQZ to be as large as possible.

Exercise 3 Vector Geometry	a	b	c	d	e	f	Points
	1.5	1.5	1	3.5	2	2	**11.5**

The plane $\mathcal{P}$, passing through the points A(-1|4|2), B(1|0|2) and C(1|-4|4), and the line ℓ, passing through the points G(2|6|7) and H(-4|9|-2) are given.

a. Determine the Cartesian equation of the plane $\mathcal{P}$.

b. Calculate the angle φ between the plane $\mathcal{P}$ and the xy-plane.

c. Find the equation of a line n, which intersects the line ℓ normally in point G.

d. Find the coordinates of the point P on the y-axis in order for the angle $\sphericalangle(\text{APB})$ to be equal to 90°.

Furthermore, the plane $\mathcal{R}$: $5x - 2y - 4z + 30 = 0$ is given.

f. Show that the plane $\mathcal{R}$ is perpendicular to the plane $\mathcal{P}$, and that the plane $\mathcal{R}$ contains the line ℓ.

g. Determine the line k of intersection between the plane $\mathcal{P}$ and $\mathcal{R}$. In what special position is the line k?

Exercise 4 Probability	a	b	c	d	e	f	g	Points
	0.5	1.5	1	1.5	1	2.5	4	**12**

The entrance area of a large leisure park has 6 cash desks, no. 1 up to no. 6. Eight employees can operate these cash desks.

a. In how many ways can the 8 employees be assigned to the 6 cash desks?

b. In how many ways is this possible if employee Fritz and employee Susi can only work at the cash desk if they are both assigned (= *eingeteilt*) together, namely Fritz at his favorite cash desk no. 2 and Susi at her favorite cash desk no. 1?

Each cash desk is chosen with the same probability. 4 families arrive and each pays at a cash desk.

c. What is the probability that all the families choose different cash desks?

A game is offered to the 18 children (10 girls and 8 boys) present in the leisure park, in which a group of four is chosen at random.

d. What is the probability that there are exactly 2 boys and 2 girls in the group of four?

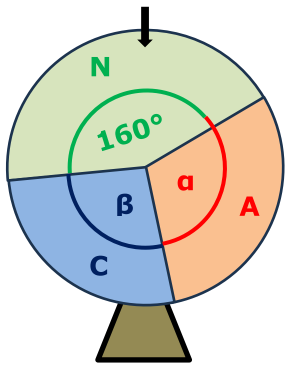
Those children who do not want to take part in the game can draw lottery tickets. Each lottery ticket has a probability of 0.05 for winning.

e. What is the probability that a child who draws 10 lottery tickets will have exactly two winning lottery tickets?

f. How many lottery tickets would a child have to draw at least in order to have at least one winning ticket with a probability of at least 0.99?

For the adults, the following game of chance is offered, in which admission cards for the leisure park can be won:

At the beginning of the game, a fair dice is rolled. If you get a "6", you can then spin a wheel of fortune with three sectors N, A and C.
If the pointer stops on sector C, you win a children's admission card worth 28 francs; if it stops on sector A, an adult admission card worth 36 francs is won. In sector N you go away empty-handed.



The center angle of sector N measures 160°. The angles α and β of sectors A and C are chosen so that an average win of 3 francs per game can be expected.

g. Calculate the central angles of sectors A and C.

Written Matura Exam 2024 — *Short Answers*

Exercise 1	a	b	c	d	**Points**
Calculus I	6	1	2	2	**11**

a. domain: $\underline{ID = \{x \in \mathbb{R} \,/\, x \neq 0\}} = \underline{\mathbb{R} \setminus \{0\}}$

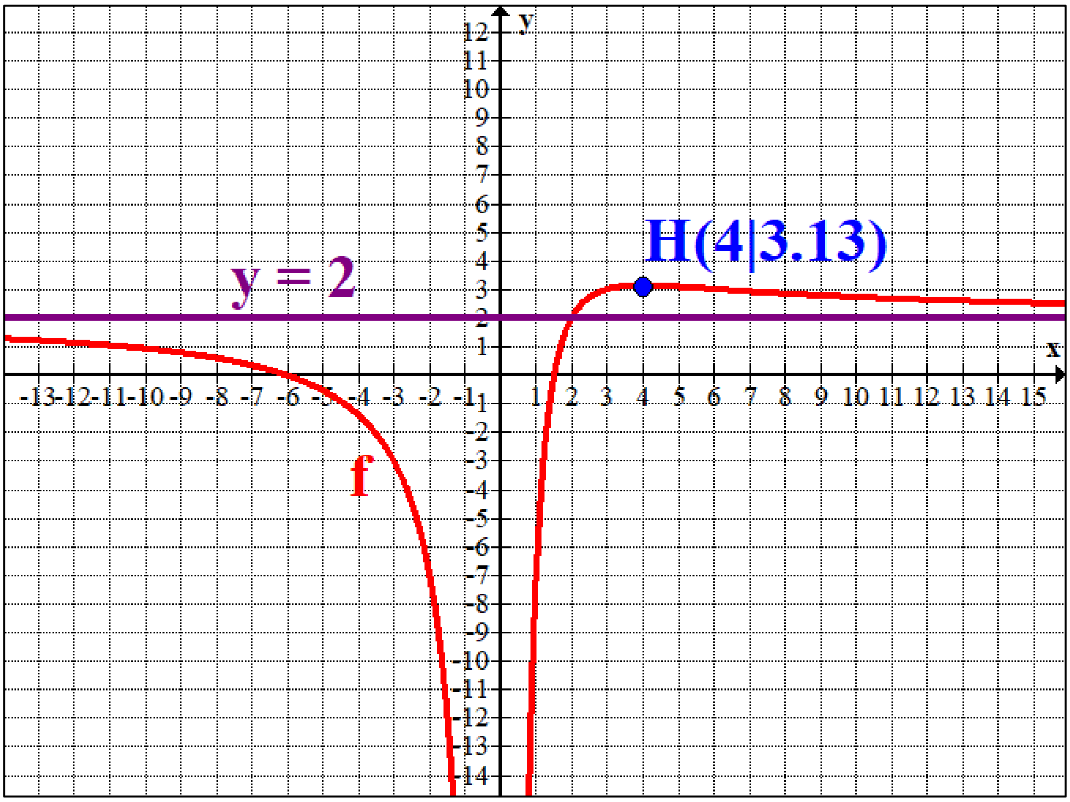
zeroes: $\underline{Z_1(-6 \mid 0), Z_2(1.5 \mid 0)}$

stat. points: $\underline{H(4 \mid 3.13)}$

behavior: <u>x = 0 is a vertical asymptote</u> with $\lim\limits_{x \to 0} \frac{2x^2 + 9x - 18}{x^2} = -\infty$

$\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \frac{2x^2 + 9x - 18}{x^2} = 2$ (due to the same highest powers in num./denom.)

<u>y = 2 is a horizontal asymptote</u>



b. Differentiate the given antiderivative to prove: F′(x) = f(x).

c. $\underline{\underline{a = 1, b = 2}}$

d. *idea*: in point P, the first derivative (= slope) of the function f is the same as value of the slope, based on the triangle of slope

tangent: $\underline{\underline{t(x) = 1.904x}}$

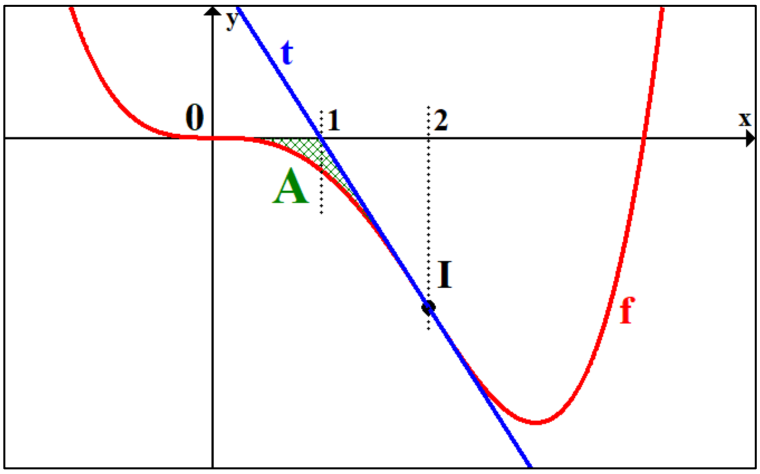
Exercise 2 **Calculus II**	a	b	c	d	**Points**
	2	1	5	4.5	**12.5**

a. $\underline{\underline{\varphi}} = \underline{\underline{81.12°}}$

b. $\underline{\underline{\dfrac{16384\pi}{1575}}} = \underline{\underline{32.68}}$

c. $\underline{t(x) = -1.6x + 1.6}$

$A = \underline{\underline{\dfrac{4}{25}}} = \underline{\underline{0.16}}$



d. <u>The triangle ΔPQZ reaches its maximum area if the point P has the coordinates P(3/8.1).</u>

Exercise 3 **Vector Geometry**	a	b	c	d	e	f	**Points**
	1.5	1.5	1	3.5	2	2	**11.5**

a. $\underline{\underline{\mathcal{P}: 2x + y + 2z - 6 = 0}}$

b. $\underline{\underline{\varphi}} = \underline{\underline{70.53°}}$

c. common point: <u>G(2|6|7)</u>

$$\underline{\underline{n: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}}$$

d. $\underline{P_1(0|1|0);\ P_2(0|3|0)}$

e. $\vec{n}_{\mathcal{P}} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\vec{n}_{\mathcal{R}} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$ with $\underline{\vec{n}_{\mathcal{P}} \bullet \vec{n}_{\mathcal{R}}} = \underline{0}$ so $\underline{\underline{\mathcal{P} \perp \mathcal{R}}}$

$G \in \mathcal{R}: \quad 10 - 12 - 28 + 30 = 0 \quad \rightarrow \quad \underline{G \in \mathcal{R}}$

$H \in \mathcal{R}: \quad -20 - 18 + 8 + 30 = 0 \quad \rightarrow \quad \underline{H \in \mathcal{R}}$

If G and H are on the plane $\mathcal{R}$, than the line through these points lies also on the plane $\mathcal{R}$.

f. idea: Find two points lying on both planes; their connection line is then the line of intersection

x = 0: there is no solution for this case → line k is parallel to the yz-plane or perpendicular to the x-axis

y = 0: $S_1(-2\,|\,0\,|\,5)$ z = 0: $S_1(-2\,|\,0\,|\,10)$

line of intersection $k: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 10 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

Exercise 4	a	b	c	d	e	f	g	**Points**
Probability	0.5	1.5	1	1.5	1.5	2	4	**12**

a. 20'160

b. 1080

c. $p = 0.28$

d. $p = \frac{7}{17} = 0.41$

e. $P[\text{exactly 2 winning tickets in 10 tickets}] = 0.07$

f. at least 90 tickets

g. The sector C has a quarter of the full angle of 360°, which means 90°, and the sector A has the remaining part, 360° - 160° - 90° = 110°.