KANTON LUZERN
Bildungs- und Kulturdepartement

Kantonsschule Alpenquai Luzern

Written Matura Exam 2021

Subject	Mathematics Basic Course
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Classes	6k, 6l
Date of the exam	Tuesday, May 25, 2021
Time	180 minutes
Aids allowed	- "Mathematics Formulary", Adrian Wetzel - A dictionary (book, no electronic translator) - TI-30X Pro (no handbook)
Instructions	- Importance is attached to a proper and clear representation. - Write each exercise on a separate sheet of paper. - All solutions must show the steps leading to the result. - Put your personal number, your name and your class on every sheet of paper.
Maximum points per exercise	Exercise 1: 11.5 Exercise 2: 11.0 Exercise 3: 8.0 Exercise 4: 3.0 Exercise 5: 11.5 **Total: 45.0** 38 points are required for a grade of 6, and 22 points are for a grade of 4.
Number of pages	5 (including title page)

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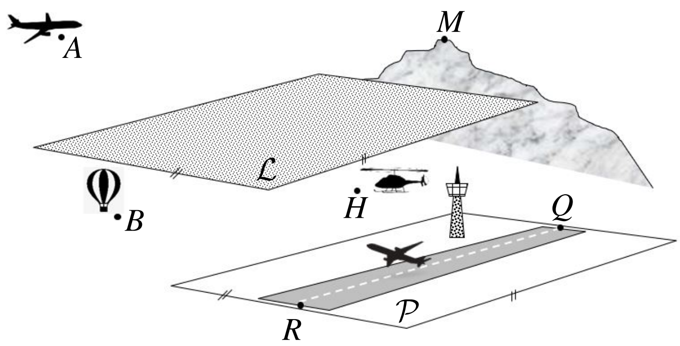
Surname, First name Class Number

	a	b	c	d	e	
Exercise 1: Vector Geometry	4.5	2	1	2	2	**11.5 points**

We consider an airport with the runway QR for starting/landing in the plane $\mathcal{P}$ close to the mountain peak M. Furthermore, there is a cloud layer [Wolkenschicht], the floor of which [deren Untergrenze] is a parallel plane $\mathcal{L}$ to the plane $\mathcal{P}$. The units of the coordinate system are in kilometers.



The points R and M are given by $R(6\,|\,4\,|\,0)$ and $M(7\,|\,2\,|\,2)$. The floor of the cloud layer can be described by the equation $\mathcal{L}: x-2y+2z-16=0$.

a) i) Calculate the distance of the aircraft, located at point $A(-6\,|\,-6\,|\,14)$, to the mountain peak M.

ii) The aircraft flies from point A in a straight line to point R for landing. At what point S does the aircraft intersect the floor of the cloud layer $\mathcal{L}$?

iii) Under what angle φ does the aircraft intersect the floor of the cloud layer $\mathcal{L}$?

b) A helicopter flies from position $H(5\,|\,9\,|\,6)$ in the direction of $\vec{v}_H = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Show that the flying routes of the aircraft, flying from point A to point R, and of the helicopter do not intersect each other.

c) Determine a Cartesian equation of the plane $\mathcal{P}$.

d) What is the distance between the layer $\mathcal{L}$ and the plane $\mathcal{P}$?

e) For a take-off, aircrafts start at $Q(2\,|\,2\,|\,0)$, accelerate in the direction of R and take off somewhere between Q and R in the direction of $\vec{v}_S = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. The space in which the aircrafts move during their take-off is called *starting space*, in which no flying object shall be present during a take-off of an aircraft.

During a take-off of an aircraft, a hot-air balloon is located at position $B(6.5\,|\,3\,|\,1)$.
Is the hot-air balloon at position B in the forbidden starting space?
Justify your answer with a calculation.

	a	b	c	d	e	f	
Exercise 2: Probability	1	1.5	1	3.5	1	3	**11 points**

The Super League is the highest level in Swiss football, where ten teams battle for the championship title. All teams play four times against each other, twice at home and twice away. Overall 180 games are played.

a) Let us consider all possible ranking lists of the top three positions of the ten teams.

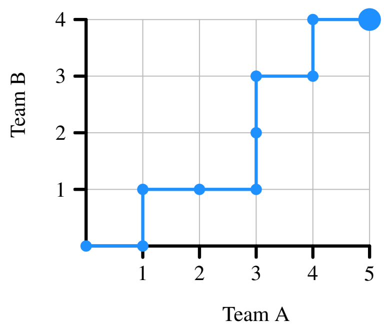
i) How many top three ranking lists are possible?

ii) How many of these ranking lists contain FC Luzern and FC Basel?

The course of a game [Spielverlauf] describes all intermediate results [Zwischenstand] of the game. A possible course of a game with the final score 5:4 is:

0:0 → 1:0 → 1:1 → 2:1 → 3:1 → 3:2 → 3:3 → 4:3 → 4:4 → 5:4

This game course is displayed in the grid.



b) i) How many possible courses of a game with a final score 5:4 exist?

ii) How many courses of a game with a final score 5:4 include the intermediate result of 3:2?

In the current season FC Luzern has 22 players in four categories: 7 attackers, 8 midfielders, 5 defenders and 2 goalkeepers.

c) All players are arranged in one line for a team picture. Players of the same category are placed next to each other. Are there more or less than 50'000'000'000 possible arrangements?

d) The coach, Fabio Celestini, puts together a team with 4 attackers, 2 midfielders, 4 defenders and 1 goalkeeper. Players of a category are chosen at random.

i) How many different teams can be put together this way?

ii) Show that the probability that the midfielder Pascal Schürpf is in such a team is 0.25.

iii) How many times at least should Fabio Celestini put a team together this way in order for the probability, that Pascal Schürpf is at least once put into a team, to be at least 95%?

There are sport statisticians who follow everything concerning football. They try to discover so-called football laws. One of these laws for the probability $P(n)$ that there are exactly n goals scored in a game is:

$$P(n) = 0.045 \cdot \frac{3.1^n}{n!}$$

Example: The probability that two goals are scored in a game is $P(2) \approx 0.216$, or 21.6%.

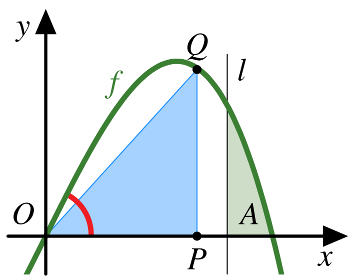
e) With the formula for $P(n)$, determine the probability that at least two goals are scored in a game.

The probability p that 5 or more goals are scored in one game is 0.203.
Each year, Mattis and Sara play the following game: If 5 or more goals per game are scored in at least 41 out of the 180 games of the Super League, Mattis gives 30 Francs to Sara. If not, Mattis receives 10 Francs from Sara.

f) Find out who profits on the long term average by playing this game.

	a	b	c	d	
Exercise 3: Calculus I	1	3	2	2	**8 points**

The figure displays the graph of $f(x) = -\frac{2}{9}x^3 + 2x$



a) Determine the angle of intersection between the graph of f and the x-axis at the origin $O(0|0)$.

b) The points P on the x-axis, Q on the graph of f and the origin O define a right-angled triangle OPQ in the first quadrant.
Find the coordinates of the point Q that maximise the size of the area of this triangle.

c) The line $l: x = a$ is parallel to the y-axis. The area A enclosed by the graph of f, the x-axis and the line l is 0.5.
Find the value of a.

d) The parabola $p(x) = -\frac{1}{6}x^2 + d$ touches the graph of f in the first quadrant.
Find the value of d.

Exercise 4: Calculus II — 3 points

A polynomial h of third order is symmetrical about the origin $O(0|0)$ and has a high point (maximum) at $H(4|16)$.
Determine the function equation of the polynomial h.

Exercise 5: Calculus III — 11.5 points

	a	b	c	d	e
Points	1.5	1	4	2	3

We consider the function $f(x) = x \cdot \sqrt{9-x^2}$

a) Show that the first derivative of f is equal to $f'(x) = \dfrac{9-2x^2}{\sqrt{9-x^2}}$

b) Investigate if the function f satisfies one of the following equations:

$$f(-x) = f(x) \qquad \text{or} \qquad f(-x) = -f(x)$$

and conclude if the graph of f features [aufzeichnen] any kind of symmetry.

c) Determine

- the domain
- the zeros
- the extrema / stationary points (The check with the second derivative is not required.)

of f and use this information to draw the graph of the function f.
Use 2 squares or 1 cm per unit.

d) Show that the function

$$F(x) = -\frac{1}{3} \cdot \left(9-x^2\right)^{\frac{3}{2}}$$

is an antiderivative of the function f and find the total area enclosed by the graph of f and the x-axis by using this antiderivative.

e) Find the equation of the tangent t to the graph of f at its middle zero and draw the tangent t as well.
For $0 \leq x \leq 3$, the area between the graph of f and the tangent t is rotated around the x-axis. Determine the volume of this solid of revolution. The integral can be determined with the calculator.

Lösungen:

Aufgabe 1:

a) a1) $\left|\overrightarrow{AM}\right| = \left|\begin{pmatrix} 7-(-6) \\ 2-(-6) \\ 2-14 \end{pmatrix}\right| = \left|\begin{pmatrix} 13 \\ 8 \\ -12 \end{pmatrix}\right|$

$= \sqrt{13^2+8^2+12^2} = \sqrt{377} = 19.42 \text{ km}$

a2) $l_{AR} : \vec{r} = \begin{pmatrix} -6 \\ -6 \\ 14 \end{pmatrix} + t \cdot \begin{pmatrix} 12 \\ 10 \\ -14 \end{pmatrix} \sim \begin{pmatrix} -6 \\ -6 \\ 14 \end{pmatrix} + t \cdot \begin{pmatrix} 6 \\ 5 \\ -7 \end{pmatrix}$

schneiden mit der Wolkengrenze $\mathcal{W}$:

$(-6+6t) - 2(-6+5t) + 2(14-7t) - 16 = 0$

$-6+6t+12-10t+28-14t-16 = 0$

$-18t + 18 = 0$

$t = 1 \quad \rightarrow \quad$ Schnittpunkt $S(0\,|-1\,|\,7)$

a3) Richtungsvektor $\vec{v}_{AR} = \begin{pmatrix} 6 \\ 5 \\ -7 \end{pmatrix}$, Normalenvektor $\vec{n}_{\mathcal{W}} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Zwischenwinkel: $\cos(\beta) = \dfrac{\vec{v}_{AR} \cdot \vec{n}_{\mathcal{N}}}{|\vec{v}_{AR}| \cdot |\vec{n}_{\mathcal{N}}|} = \dfrac{6-10-14}{\sqrt{110} \cdot 3} = -\dfrac{6}{\sqrt{110}}$

$\rightarrow \quad \beta \approx 124.90°$

Winkel zwischen Flugbahn und Wolkengrenze: $\varphi = \beta - 90° \approx 34.90°$

b) Flugbahn Flugzeug: $l_{AR} : \vec{r} = \begin{pmatrix} -6 \\ -6 \\ 14 \end{pmatrix} + t \cdot \begin{pmatrix} 6 \\ 5 \\ -7 \end{pmatrix}$

Flugbahn Helikopter: $l_H : \vec{r} = \begin{pmatrix} 5 \\ 9 \\ 6 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

schneiden von l_{AR} und l_H: $\begin{array}{lrcl} (1) & -6+6t & = & 5+s \\ (2) & -6+5t & = & 9+2s \\ (3) & 14-7t & = & 6-s \end{array}$

Gleichungen (1) und (2) werden durch $s = -5$ und $t = 1$ gelöst.

Einsetzen in (3): $\quad 14-7 \neq 6+5 \quad \Rightarrow \quad$ Flugbahnen schneiden sich nicht

c) Ebene $\mathcal{E}$ parallel zur Wolkengrenze $\mathcal{W}$: $\quad \mathcal{E} : x - 2y + 2z + D = 0$

R einsetzen: $6 - 8 + 0 + D = 0 \quad \rightarrow \quad D = 2 \quad \rightarrow \quad \mathcal{P} : x - 2y + 2z + 2 = 0$

d) Lot auf $\mathcal{W}$ durch R: $\quad n : \vec{r} = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$n \cap \mathcal{W}$: $\quad (6+t) - 2(4-2t) + 2(2t) - 16 = 9t - 18 = 0 \quad \rightarrow \quad t = 2$

Punkt U auf Wolkengrenze: $\quad U(8\,|\,0\,|\,4)$

Gesuchter Abstand: $\overline{RU} = \left| \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} \right| = \sqrt{2^2 + 4^2 + 4^2} = 6$ km

e) Der Startraum S ist eine Ebene mit Stützpunkt $Q(2\,|\,2\,|\,0)$ und den Richtungsvektoren $\overrightarrow{QR} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ und $\vec{v}_S = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

Lösung mit Parametergleichung:

$$S : \vec{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

B in S? $\quad \begin{pmatrix} 6.5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Aus Gleichung für z-Komponente: $t = 0.5$

Aus Gleichung für y-Komponente: $3 = 2 + 2s - 0.5 \quad \rightarrow \quad s = 0.75$

Überprüfen mit Gleichung für x-Komponente: $6.5 = 2 + 3 + 1.5$ ✓

B liegt im verbotenen Startraum.

Lösung mit Koordinatengleichung:

$$S : \vec{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Gleichungssystem: $\begin{array}{rcl} x &=& 2 + 2s + 3t \\ y &=& 2 + s - t \\ z &=& 2t \end{array} \quad \rightarrow \quad \begin{array}{rcl} x &=& 2 + 2s + 3t \\ t &=& \frac{z}{2} \\ s &=& y - 2 + t = y - 2 + \frac{z}{2} \end{array}$

$x = 2 + 2(y - 2 + \frac{z}{2}) + \frac{3z}{2} = 2y + \frac{5z}{2} - 2 \quad \rightarrow \quad S : 2x - 4y - 5z + 4 = 0$

$B \in S$? $\quad 13 - 12 - 5 + 4 = 0$ ✓

B liegt im verbotenen Startraum.

Lösung mit Flugbahn des Flugzeugs:

Flugzeug bewegt sich auf der Flugbahn $\quad \vec{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Testen, ob B auf der Flugbahn: $\begin{pmatrix} 6.5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

rechnerisch wie erster Lösungsweg

Aufgabe 2:

a) a1) Anzahl Ranglisten: $10 \cdot 9 \cdot 8 = 720$

a2) Anzahl Listen: $8 \cdot 3! = 48$

b) b1) Anzahl Spielverläufe: $\binom{9}{5} = 126$

b2) Anzahl Spielverläufe: $\binom{5}{2} \cdot \binom{4}{2} = 60$

c) Anzahl Anordnungen: $4! \cdot (7! \cdot 8! \cdot 5! \cdot 2!) \approx 1.17 \cdot 10^{12}$

$1.17 \cdot 10^{12}$ ist grösser als $5 \cdot 10^{10}$.

d) d1) Anzahl Teams: $\binom{7}{4} \cdot \binom{8}{2} \cdot \binom{5}{4} \cdot \binom{2}{1} = 9800$

d2) W'keit, Pascal Schürpf auszuwählen: $\dfrac{\binom{7}{1}}{\binom{8}{2}} = \frac{1}{4}$

d3) n : Anzahl zusammengestellte Teams

$$\begin{aligned}
P(\text{Schürpf in mind. einem Team}) &\geq 0.95 \\
1 - P(\text{Schürpf in keinem Team}) &\geq 0.95 \\
1 - \left(\frac{3}{4}\right)^n &\geq 0.95 \\
\left(\frac{3}{4}\right)^n &\leq 0.05 \\
n \cdot \ln\left(\frac{3}{4}\right) &\leq \ln(0.05) \\
n &\geq \frac{\ln(0.05)}{\ln\left(\frac{3}{4}\right)} \approx 10.4
\end{aligned}$$

$\rightarrow$ mindestens 11 Zusammenstellungen

e) $P(\text{mind. 2 Tore}) = 1 - P(0 \text{ Tore}) - P(1 \text{ Tor}) = 1 - 0.045 \cdot \frac{1}{1} - 0.045 \cdot \frac{3.1}{1} = 0.8155$

f) Binomialverteilung mit $n = 180$ und $p = 0.203$

$P(k \leq 40) = \sum_{k=0}^{40} \binom{180}{k} \cdot 0.203^k \cdot 0.797^{180-k} \approx 0.771$

$P(k \geq 41) = 1 - P(k \leq 40) \approx 1 - 0.771 = 0.229$

Erwartungswert für Mattis: $E \approx 0.771 \cdot 10 + 0.229 \cdot (-30) \approx 0.84$

Das Spiel lohnt sich für Mattis.

Aufgabe 3:

a) $f'(x) = -\frac{2}{3}x^2 + 2 \qquad f'(0) = 2$

Schnittwinkel: $\varphi = \arctan(2) \approx 63.43°$

b) $A(u) = \frac{1}{2} \cdot \overline{OP} \cdot \overline{PQ} = \frac{1}{2} \cdot u \cdot f(u) = \frac{1}{2} \cdot u \cdot \left(-\frac{2}{9}u^3 + 2u\right) = -\frac{1}{9}u^4 + u^2$

$A'(u) = -\frac{4}{9}u^3 + 2u$

Optimierung: $-\frac{4}{9}u^3 + 2u = 0 \quad \Leftrightarrow \quad (u_1 = -\frac{3\sqrt{2}}{2}),\ (u_2 = 0),\ u_3 = \frac{3\sqrt{2}}{2}$

$A(u_3)$ muss ein Maximum sein, da $A(u_2) = 0$ und $A(u_3) > 0$.

$Q(\frac{3\sqrt{2}}{2} \,|\, \frac{3\sqrt{2}}{2}) \approx (2.12 \,|\, 2.12)$

c) Nullstellen von f:

$f(x) = 0 \quad \Leftrightarrow \quad x(-\frac{2}{9}x^2 + 2) = 0 \quad \Leftrightarrow \quad x = 0$ oder $-\frac{2}{9}x^2 + 2 = 0$

$\Leftrightarrow \quad x = 0$ oder $\frac{2}{9}x^2 = 2 \quad \Leftrightarrow \quad x = 0$ oder $x^2 = 9 \quad \Leftrightarrow \quad x = 0$ oder $x = \pm 3$

$F = \int_a^3 f(x)\,dx = \int_a^3 (-\frac{2}{9}x^3 + 2x)\,dx = \left(-\frac{1}{18}x^4 + x^2\right)\Big|_a^3$

$= -\frac{1}{18} \cdot 3^4 + 3^2 + \frac{1}{18}a^4 - a^2 = \frac{1}{18}a^4 - a^2 + \frac{9}{2} = \frac{1}{2}$

$\Leftrightarrow \quad a^4 - 18a^2 + 72 = 0$

Die Lösung a mit $0 < a < 3$ ist $a = \sqrt{6} \approx 2.45$

d) Berührpunkt sei $B(u \,|\, v)$. Er ist auch Schnittpunkt.

$p'(u) = f'(u) \quad \Leftrightarrow \quad -\frac{1}{3}u = -\frac{2}{3}u^2 + 2 \quad \Leftrightarrow \quad \frac{2}{3}u^2 - \frac{1}{3}u - 2 = 0$

$\Leftrightarrow \quad (u_1 = -\frac{3}{2}),\ u_2 = 2$

$p(u) = f(u) \quad \Leftrightarrow \quad -\frac{1}{6}u^2 + d = -\frac{2}{9}u^3 + 2u \quad \Leftrightarrow \quad d = -\frac{2}{9}u^3 + \frac{1}{6}u^2 + 2u$

mit $u = 2$: $\qquad d = \frac{26}{9}$

Aufgabe 4:

Da h punktsymmetrisch bzgl. $(0 \,|\, 0)$: $\qquad h(x) = ax^3 + cx$

$h(4) = 64a + 4c = 16$

$h'(4) = 48a + c = 0$

GS lösen: $a = -\frac{1}{8}$, $c = 6 \quad \rightarrow \quad h(x) = -\frac{1}{8}x^3 + 6x$

Aufgabe 5:

a) Unter Verwendung von Produkt- und Kettenregel:

$$\begin{aligned} f'(x) &= (x)' \cdot \sqrt{9-x^2} + x \cdot \left(\sqrt{9-x^2}\right)' \\ &= 1 \cdot \sqrt{9-x^2} + x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{9-x^2}} \cdot \left(9-x^2\right)' \\ &= \sqrt{9-x^2} + x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{9-x^2}} \cdot (-2x) \\ &= \sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}} \\ &= \frac{9-x^2}{\sqrt{9-x^2}} - \frac{x^2}{\sqrt{9-x^2}} = \frac{9-2x^2}{\sqrt{9-x^2}} \end{aligned}$$

b) $f(-x) = (-x) \cdot \sqrt{9-x^2} = -x \cdot \sqrt{9-x^2} = -f(x)$

$\rightarrow$ punktsymmetrisch zum Ursprung

c) $D = [-3, 3]$

Nullstellen: -3, 0 und 3

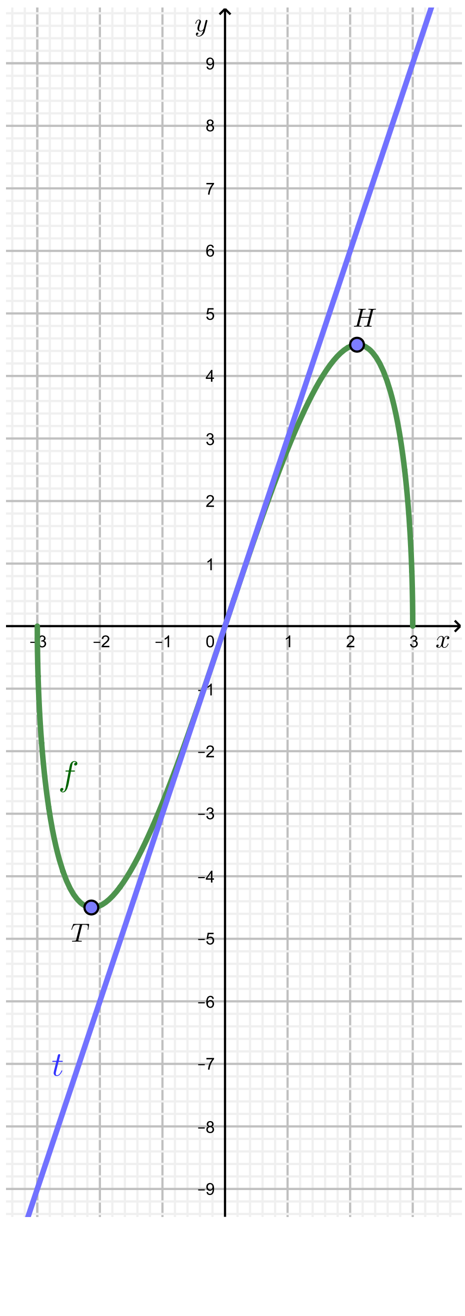
$f'(x) = \dfrac{9-2x^2}{\sqrt{9-x^2}} = 0 \quad \Rightarrow 9 - 2x^2 = 0$

$\Rightarrow x_E = \pm\sqrt{\dfrac{9}{2}} = \pm\dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2} \approx \pm 2.121$

$f(x_E) = \pm\dfrac{3}{\sqrt{2}} \cdot \sqrt{9 - \dfrac{9}{2}} = \pm\dfrac{3}{\sqrt{2}} \cdot \dfrac{3}{\sqrt{2}} = \pm\dfrac{9}{2}$

Extremalpunkte: $H(2.121\,|\,4.5)$, $T(-2.121\,|\,-4.5)$

Graph: rechts



d) $F'(x) = -\frac{1}{3} \cdot \frac{3}{2} \cdot (9-x^2)^{1/2} \cdot (-2x)$

$\quad = x \cdot (9-x^2)^{1/2} = f(x)$

Symmetrie von f kann ausgenutzt werden:

$A = 2 \cdot \int_0^3 f(x)\,dx = 2 \cdot (F(3) - F(0))$

$\quad = \frac{2}{3} \cdot 9^{3/2} = 18$

e) $f'(0) = \frac{9}{\sqrt{9}} = \frac{9}{3} = 3$

Tangente t: $\quad y - f(0) = 3 \cdot (x - 0) \quad \Rightarrow \quad y = 3x$

$$V_{\text{aussen}} = \pi \int_0^3 (t(x))^2 \; dx = \pi \int_0^3 (3x)^2 \, dx = 9\pi \int_0^3 x^2 \, dx = 9\pi \cdot \frac{1}{3} x^3 \bigg|_0^3 = 81\pi$$

$$V_{\text{innen}} = \pi \int_0^3 (f(x))^2 \; dx = \pi \int_0^3 x^2 \cdot (9 - x^2) \, dx = \pi \int_0^3 (9x^2 - x^4) \, dx$$

$$= \pi \left(9 \cdot \frac{1}{3} x^3 - \frac{1}{5} x^5 \right) \bigg|_0^3 = \pi \left(81 - \frac{243}{5} \right) = \frac{162}{5} \pi$$

$$V = V_{\text{aussen}} - V_{\text{innen}} = \left(81 - \tfrac{162}{5} \right) \pi = \tfrac{243}{5} \pi \approx 152.68$$