KANTON LUZERN
Bildungs- und Kulturdepartement

Kantonsschule Alpenquai Luzern

Written Matura Exam 2019

Subject	Mathematics Basic Course
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Classes	6i / 6k
Date of the exam	May 24, 2019
Time	180 minutes
Aids allowed	- "Mathematics Formulary", Adrian Wetzel - A dictionary (book, no electronic translator) - TI-30X Pro
Instructions	- Importance is attached to a proper and clear representation. - Write each exercise on a separate sheet of paper. - All solutions must show the steps leading to the result. - Put your personal number, your name and your class on every sheet of paper.
Maximum points per exercise	Exercise 1: 11 Exercise 2: 10 Exercise 3: 12.5 Exercise 4: 11.5 **Total: 45** 40 points are required for a grade of 6.
Number of pages	5 (including title page)

....
Surname, First name Class Number

Exercise 1 Vector Geometry	a	b	c	d	e	f	g	Points
	2	1	1	1	2	2	2	11

Given are the points $A(10\,|\,5\,|\,13)$, $B(6\,|\,9\,|\,-3)$ and $M(2\,|\,1\,|\,5)$, as well as the plane $\mathcal{P}_1$: $13x + 29y + 4z - 327 = 0$.

a. Determine the Cartesian equation of the plane $\mathcal{P}_2$ that contains the points A, B and M.

b. Show that the triangle ABM is right-angled and isosceles, with the right angle at M.

c. Determine the coordinates of the points C and D so that the quadrilateral ABCD is a square with midpoint M.

d. Show that the points A and B lie in the plane $\mathcal{P}_1$.

For the further exercises, the normal vector $\overrightarrow{n_{\mathcal{P}_2}} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ of the plane $\mathcal{P}_2$ is given.

e. The point S lies in the plane $\mathcal{P}_1$ and it is the apex (=Spitze) of the right square based pyramid ABCDS. Calculate the coordinates of S.

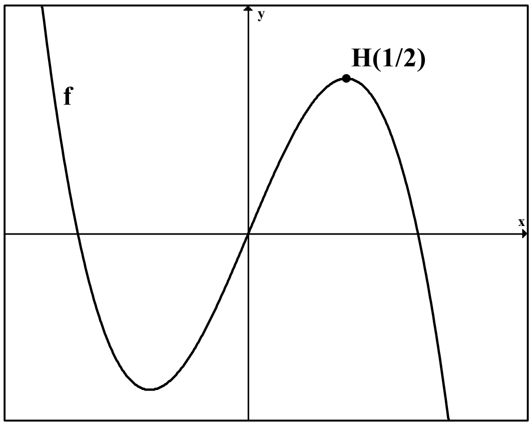
f. Calculate the angle between the base ABCD and the lateral area ABS of the pyramid. *Hint*: The lateral area ABS of the pyramid lies in $\mathcal{P}_1$.

g. Calculate the volume of the pyramid ABCDS.

If you could not calculate the coordinates of the apex S of the pyramid in subtask e., use the substitute point $\hat{S}(16\,|\,-13\,|\,-2)$. $\hat{S}$ defines with the square ABCD a right square based pyramid as well.

Exercise 2	a	b	c	d	e	**Points**
Calculus	1	2	2	2	3	10

We consider the function $f(x) = -x^3 + 3x$ (*see graph at the right*).



a. Find the x-coordinates of the points on the graph of $f(x)$ where the graph has a slope of $m = -24$.

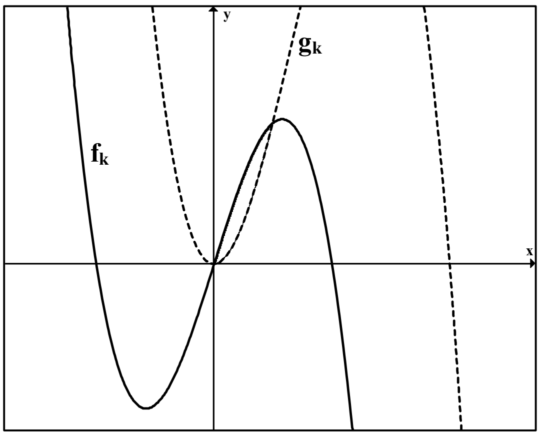
b. Find the equation of the tangent of inflection to the graph of $f(x)$.

c. Determine the area of the region enclosed by the graph of $f(x)$ and the tangent in the high point H(1/2) of the graph of $f(x)$.

Now, we consider the more general function $f_k(x) = -x^3 + kx$ (*the function above was the special case with k = 3*).

d. Show that the y-intercept of the tangent t in point $P(1 / y_P)$ to the graph of $f_k(x)$ does not depend on k by calculating the equation of tangent t.

e. The graphs of the functions $f_k(x)$ and $g_k(x) = -x^3 + kx^2$ with k > 0 enclose a region in the first quadrant. Find the value of k in order for the area of this region to measure $A = \frac{2}{3}$.



Exercise 3	a	b	c	d	e	f	Points
Calculus	3.5	2	1.5	1	1.5	3	12.5

The function $f(x) = (1 - x) \cdot e^x$ and its first two derivatives are given:

$$f'(x) = -x \cdot e^x \qquad f''(x) = -(x+1) \cdot e^x .$$

a. Determine the zeros, maximum and minimum points (= extrema) and the asymptotes of $f(x)$ and then draw the graph of the function $f(x)$. *Units: 2 squares or 1cm.*

b. Find the equation of a quadratic function $g(x) = ax^2 + bx + c$ which has at $x = 0$ the same function value, the same value for the first derivative and the same value for the second derivative as $f(x)$.

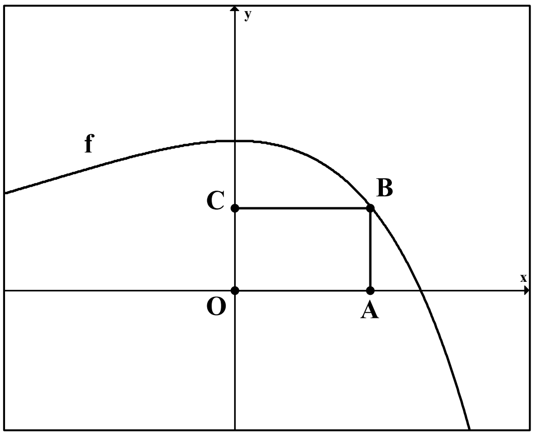
If you could not find the equation of function $g(x)$, continue with $g(x) = -\frac{1}{3} \cdot x^2 + 1$.

c. The part of the graph of $g(x)$ lying above the x-axis is rotating across the x-axis. Calculate the volume of this solid of revolution.

d. Show that the function $F(x) = (2 - x) \cdot e^x$ is an antiderivative of $f(x)$.

e. The curve $f(x)$, the negative x-axis and the positive y-axis enclose a region which stretches to infinity to the left. Using the antiderivative from exercise d., calculate the area of this region.

f. The curve $f(x)$, the positive x-axis and the positive y-axis enclose a region. Into this region, the rectangle OABC is inscribed where O(0/0) is the origin, A lies on the positive x-axis, B on the graph of $f(x)$ and C on the positive y-axis (*see sketch at the right*). What are the coordinates of point A in order for the area of the rectangle OABC to be as large as possible?
The check of the maximum by the second derivative is not required.



<table>
<tr><td rowspan="4">Exercise 4
Probability</td><td>a_1</td><td>a_2</td><td>b_1</td><td>b_2</td><td rowspan="2">Points</td></tr>
<tr><td>0.5</td><td>1</td><td>0.5</td><td>1.5</td></tr>
<tr><td>b_3</td><td>c</td><td>d</td><td>e</td><td rowspan="2">11.5</td></tr>
<tr><td>1.5</td><td>2.5</td><td>2</td><td>2</td></tr>
</table>

At a fairground a game with three buckets (= Eimer) is offered. The task of this game is to throw a ball into these buckets. Throwing the ball into the biggest bucket is rewarded with one point, throwing the ball into the medium sized bucket is rewarded with four points and throwing the ball into the smallest bucket results in nine points. Otherwise, no points are given. Each ball that is thrown costs 1 Swiss Franc.

Chiara repeatedly buys five balls and throws them. Her friend Daniel writes down the outcome of five throws as follows: «9 0 9 4 1», «0 4 9 9 1» or «9 0 9 1 0».

a. How many different outcomes of five throws are there
   - a_1. in general?
   - a_2. if exactly two balls are thrown into the smallest bucket?

The probability that Chiara throws a ball into the biggest bucket is 30%, the probability, that she throws it into the medium bucket is 20% and the probability that she throws it into the smallest bucket is 10%.

b. Chiara throws five balls. What is the probability that
   - b_1. no ball ends up in a bucket?
   - b_2. exactly one ball ends up in a bucket?
   - b_3. at least three balls end up in buckets?

c. How many balls would Chiara have to buy at least, so that the probability is at least 95% that she will throw a ball into the smallest bucket at least once?

Depending on the number of points achieved, different prizes can be won. The gigantic Teddy Bear is given away for 100 points.

d. How much does it cost Chiara on average to win the gigantic Teddy Bear?

At a fundraising event, the game is offered slightly altered. The balls may be thrown repeatedly until they end up in a bucket.

e. What is now the probability that Chiara throws all five balls into the same bucket?

Schriftliche Maturitätsprüfung 2019

Lösungen

Aufgabe 1

a.

$$\overrightarrow{MA}=\begin{bmatrix}8\\4\\8\end{bmatrix},\quad \overrightarrow{MB}=\begin{bmatrix}4\\8\\-8\end{bmatrix},\quad \overrightarrow{MA}\times\overrightarrow{MB}=\begin{bmatrix}-96\\96\\48\end{bmatrix}=-48\cdot\begin{bmatrix}2\\-2\\-1\end{bmatrix}$$

$$F:\quad 2x-2y-z+d=0$$
$$M\in F:\quad 2\cdot 2-2\cdot 1-5+d=0 \quad\rightarrow\quad d=3$$
$$F:\quad 2x-2y-z+3=0$$

Oder

$$F:\quad \vec{r}=\begin{bmatrix}2\\1\\5\end{bmatrix}+t\cdot\begin{bmatrix}2\\1\\2\end{bmatrix}+s\cdot\begin{bmatrix}1\\2\\-2\end{bmatrix}\qquad \begin{aligned}x&=2+2t+s\\ y&=1+t+2s\\ z&=5+2t-2s\end{aligned}\qquad \begin{aligned}y-2x&=-3-3t\\ y+z&=6+3t\end{aligned}$$

$$-2x+2y+z=3 \qquad F:\quad 2x-2y-z+3=0$$

b.

$$\overline{MA}=\overline{MB}=12,\quad \overrightarrow{MA}\cdot\overrightarrow{MB}=0 \quad / \quad \overline{AB}=12\sqrt{2}=\sqrt{12^2+12^2}$$

c.

$$\overrightarrow{r_C}=\overrightarrow{r_M}-\overrightarrow{MA}=\begin{bmatrix}2\\1\\5\end{bmatrix}-\begin{bmatrix}8\\4\\8\end{bmatrix}=\begin{bmatrix}-6\\-3\\-3\end{bmatrix}\qquad \overrightarrow{r_D}=\overrightarrow{r_M}-\overrightarrow{MB}=\begin{bmatrix}2\\1\\5\end{bmatrix}-\begin{bmatrix}4\\8\\-8\end{bmatrix}=\begin{bmatrix}-2\\-7\\13\end{bmatrix}$$

$$C\left(-6\,|-3\,|-3\right),\ D\left(-2\,|-7\,|\,13\right)$$

d.

$$13\cdot 10+29\cdot 5+4\cdot 13-327=0 \quad\rightarrow\quad A\in E$$
$$13\cdot 6+29\cdot 9+4\cdot(-3)-327=0 \quad\rightarrow\quad B\in E$$

e.

$$h:\quad \vec{r}=\begin{bmatrix}2\\1\\5\end{bmatrix}+t\cdot\begin{bmatrix}2\\-2\\-1\end{bmatrix}$$

$$\{S\}=h\cap E:\quad 13\cdot(2+2t)+29\cdot(1-2t)+4\cdot(5-t)-327=0$$
$$-36t-252=0 \quad\rightarrow\quad t=-7 \quad\rightarrow\quad S\left(-12\,|\,15\,|\,12\right)$$

f.

$$\overrightarrow{n_E} = \begin{bmatrix} 13 \\ 29 \\ 4 \end{bmatrix}, \quad \overrightarrow{n_F} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix},$$

$$\cos(\varphi) = \frac{\left|\overrightarrow{n_E} \cdot \overrightarrow{n_F}\right|}{\left|\overrightarrow{n_E}\right| \cdot \left|\overrightarrow{n_F}\right|} = \frac{\left|13 \cdot 2 + 29 \cdot (-2) + 4 \cdot (-1)\right|}{\sqrt{13^2 + 29^2 + 4^2} \cdot \sqrt{2^2 + (-2)^2 + (-1)^2}} = \frac{|-36|}{\sqrt{1026 \cdot 3}} \approx 0.3746,$$

$$\varphi \approx 67.998°$$

g.

$$\left|\overrightarrow{MS}\right| = 21, \quad \left|\overrightarrow{AB}\right| = 12\sqrt{2}, \quad V = \tfrac{1}{3} \cdot \left|\overrightarrow{AB}\right|^2 \cdot \left|\overrightarrow{MS}\right| = \tfrac{1}{3} \cdot 288 \cdot 21 = 2016$$

Oder:

$$V = 4 \cdot \tfrac{1}{6} \cdot \left|\left(\overrightarrow{MA} \times \overrightarrow{MB}\right) \cdot \overrightarrow{MS}\right| = \tfrac{2}{3} \cdot \left\| \begin{bmatrix} -96 \\ 96 \\ 48 \end{bmatrix} \cdot \begin{bmatrix} -14 \\ 14 \\ 7 \end{bmatrix} \right\| = 2016$$

$$V = \tfrac{1}{3} \cdot \left|\left(\overrightarrow{BA} \times \overrightarrow{BC}\right) \cdot \overrightarrow{MS}\right| = \left\| \left(\begin{bmatrix} 4 \\ -4 \\ 16 \end{bmatrix} \times \begin{bmatrix} -12 \\ -12 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} -14 \\ 14 \\ 7 \end{bmatrix} \right\| = \tfrac{1}{3} \cdot \left\| \begin{bmatrix} 192 \\ -192 \\ -96 \end{bmatrix} \cdot \begin{bmatrix} -14 \\ 14 \\ 7 \end{bmatrix} \right\| = 2016$$

Mit Ersatzpunkt: gleiches Volumen.

Aufgabe 2

a. $f(x) = -x^3 + 3x \rightarrow f'(x) = -3x^2 + 3 = -24 \rightarrow -3x^2 = -27 \rightarrow x^2 = 9 \rightarrow \underline{\underline{x = \pm 3}}$

b. Wendepunkt: $f(x) = -x^3 + 3x \quad f'(x) = -3x^2 + 3 \quad f''(x) = -6x = 0 \rightarrow \underline{x = 0}$

überprüfe: $f'''(x) = -6 \neq 0 \rightarrow \underline{W(0/0)}$

Wendetangente: $m_t = f'(0) = 3 \rightarrow \underline{w(x) = 3x}$

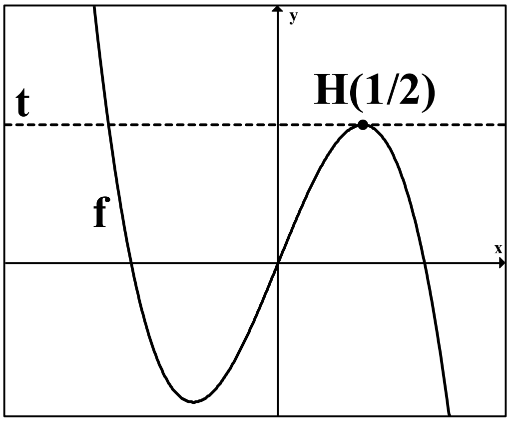
c. Tangente in H: $t(x) = 2$

Schnittpunkt von t und f: $x = -2$

$$A = \int_{-2}^{1} \left(t(x) - f(x)\right)dx$$

$$= \int_{-2}^{1} \left(2 + x^3 - 3x\right)dx$$

$$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1} = \underline{\frac{27}{4}}$$



d. $f_k(x) = -x^3 + kx \rightarrow f_k'(x) = -3x^2 + k$ mit $\underline{m_t} = f_k'(1) = -3(1)^2 + k = \underline{k-3}$

$\underline{y_p} = f_k(1) = -(1)^3 + k \cdot (1) = \underline{k-1} \rightarrow \underline{P(1/k-1)}$

$k - 1 = t(1) = m_t \cdot x + q = (k-3)(1) + q = k - 3 + q \rightarrow \underline{q = 2} \rightarrow \underline{\underline{t(x) = (k-3)x + 2}}$

Die Tangente schneidet die y-Achse an der Stelle $q = 2$.

e. Schnittpunkte: $f_k(x) = g_k(x) \rightarrow -x^3 + kx = -x^3 + kx^2 \rightarrow \not{k}x = \not{k}x^2$

$x = x^2 \rightarrow \underline{x = 0;1}$

Eingeschl. Fläche: $A = \int_0^1 \left(f_k(x) - g_k(x)\right)dx = \int_0^1 \left(\cancel{-x^3} + kx + \cancel{x^3} - kx^2\right)dx$

$$= k \cdot \int_0^1 \left(x - x^2\right)dx = k \cdot \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = k \cdot \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{k}{6} = \frac{2}{3} \rightarrow \underline{\underline{k = 4}}$$

Aufgabe 3

a. Nullstellen: $f(x)=(1-x)\cdot e^{x}=0 \rightarrow x=1 \rightarrow \underline{N(1/0)}$

Extrema: $f'(x)=-x\cdot e^{x}=0 \rightarrow \underline{x=0}$

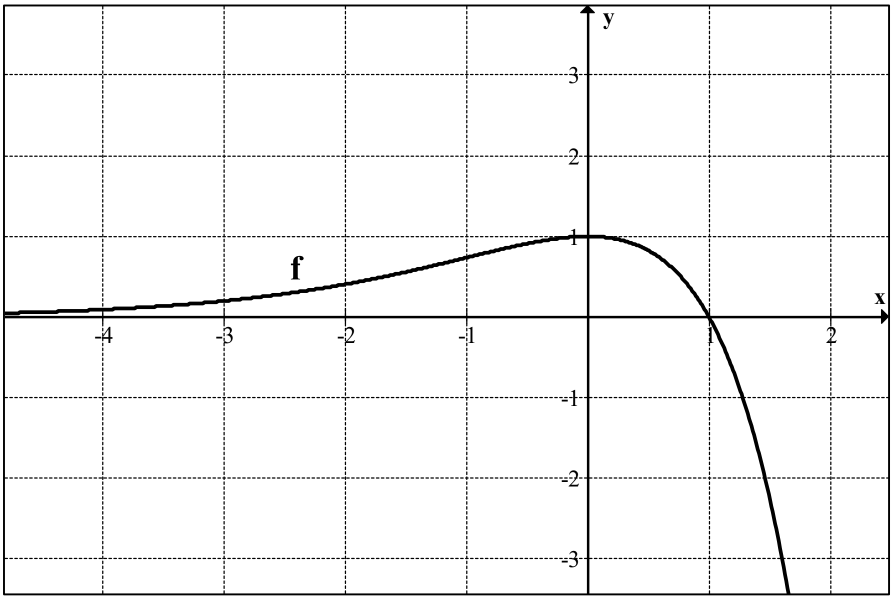
Nachweis: $f''(0)=-(0+1)\cdot e^{0}=-1 \rightarrow Max \rightarrow \underline{H(0/1)}$

Asymptoten: keine vertikalen bzw. schiefe Asymptoten

$\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\left((1-x)\cdot e^{x}\right)=-\infty \rightarrow$ keine Asymptote

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\left((1-x)\cdot e^{x}\right)=0 \rightarrow \underline{y=0}$ als horizontale Asymptote

Graph:



b. $g(x)=ax^2+bx+c$ $g'(x)=2ax+b$ $g''(x)=2a$

Bedingungen:

$g(0)=f(0) \rightarrow c=1$

$g'(0)=f'(0) \rightarrow b=0$

$g''(0)=f''(0) \rightarrow 2a=-1 \rightarrow a=-\frac{1}{2}$

$\underline{\underline{g(x)}}=-\frac{1}{2}x^2+1$

c. $g(x) = -\frac{1}{2}x^2 + 1 = -\frac{1}{2}\left(x^2 - 2\right) = 0 \rightarrow \underline{x = \pm\sqrt{2}}$

$$V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} g^2(x)dx = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(-\frac{x^2}{2}+1\right)^2 dx = 2\pi \int_0^{\sqrt{2}} \left(-\frac{x^2}{2}+1\right)^2 dx = \frac{2\pi}{4} \int_0^{\sqrt{2}} \left(x^4 - 4x^2 + 4\right)dx$$

$$= \frac{\pi}{2}\left[\frac{x^5}{5} - \frac{4x^3}{3} + 4x\right]_0^{\sqrt{2}} = \frac{\pi}{2}\left(\frac{4\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} + 4\sqrt{2} - 0\right) = \underline{\underline{\frac{16\sqrt{2}\,\pi}{15}}} \approx 4.7391$$

Mit Ersatzfunktion:

$g(x) = -\frac{1}{3}x^2 + 1 = -\frac{1}{3}\left(x^2 - 3\right) = 0 \rightarrow \underline{x = \pm\sqrt{3}}$

$$V = \pi \int_{-\sqrt{3}}^{\sqrt{3}} g^2(x)dx = \pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(-\frac{x^2}{3}+1\right)^2 dx = 2\pi \int_0^{\sqrt{3}} \left(-\frac{x^2}{3}+1\right)^2 dx = \underline{\underline{\frac{16\sqrt{3}\,\pi}{15}}} \approx 5.8042$$

d. $\underline{\underline{F'(x)}} = \frac{d}{dx}\left[\left(2-x\right)\cdot e^x\right] = \left(-1\right)\cdot e^x + \left(2-x\right)\cdot e^x = -e^x + 2e^x - x\cdot e^x = e^x - x\cdot e^x = \underline{\underline{\left(1-x\right)\cdot e^x}}$

e. $\underline{\underline{A}} = \int_{-\infty}^0 f(x)dx = \lim_{u\to-\infty} \int_u^0 f(x)dx = \lim_{u\to-\infty} \left[\left(2-x\right)\cdot e^x\right]_u^0 = \lim_{u\to-\infty} \left(2\cdot e^0 - \left(2-u\right)\cdot e^u\right) = \underline{\underline{2}}$

f. Koordinaten von A(u/0)

Zielfunktion (ZF) Fläche = $A = \overline{OA}\cdot\overline{AB} = u\cdot f(u)$

Nebenbedingung (NB): nicht nötig

Angepasste ZF: $A(u) = u\cdot f(u) = u\cdot\left(1-u\right)\cdot e^u = \underline{\left(u-u^2\right)\cdot e^u}$

$$A'(u) = \left(1-2u\right)\cdot e^u + \left(u-u^2\right)\cdot e^u = \left(1-u-u^2\right)\cdot e^u = 0$$

$$\rightarrow 1-u-u^2 = 0 \rightarrow \underline{u = 0.62;\ \cancel{-1.62}}$$

<u>Für den Punkt A(0.62/0) besitzt das Rechteck OABC die maximale Fläche.</u>

Aufgabe 4 Wahrscheinlichkeitsrechnung

a. a1. $4^5 = 1024$

a2. $\binom{5}{2} \cdot 3^3 = 270$

b. b1. $P(\text{in kein Eimer wird getroffen}) = (0.4)^5 = 1.024\%$

b2. $P(\text{nur genau einmal in Eimer treffen}) = \binom{5}{1}\ 0.6 \cdot (0.4)^4 = 7.68\%$

b3. $P(\text{mindestens drei Mal treffen}) = \sum_{k=3}^{5} \binom{5}{k}\ 0.6^k \cdot 0.4^{5-k} = 68.256\%$

c. $P(\text{mind. einmal den kleinen Eimer treffen}) \geq 0.95$ mit dem Gegenereignis operieren:

$P(\text{nie den kleinen Eimer treffen}) < 0.05$

$$(0.9)^n < 0.05 \qquad |\ln$$

$$n \cdot \ln(0.9) < \ln(0.05)$$

$$n > \frac{\ln(0.05)}{\ln(0.9)} \approx 28.43 \Rightarrow 29 \text{ Bälle}$$

d. X_i : Punktzahl bei Ereignis i $\qquad$ p_i : Wahrscheinlichkeit von Ereignis i

$$E(X) = \sum_{i=1}^{4} p_i \cdot X_i = 0.4 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 4 + 0.1 \cdot 9 = 2$$

Der Erwartungswert bei diesem Spiel ist 2 Punkte. Das heisst, man müsste 50 Mal spielen, das heisst im Schnitt kostet es 50 Franken, den Hauptpreis zu gewinnen.

e. Neue Wahrscheinlichkeiten: $P(\text{kleinen Eimer treffen}) = 0.1\overline{6} = \frac{1}{6}$,

$P(\text{mittleren Eimer treffen}) = 0.\overline{3} = \frac{1}{3}$,

$P(\text{grossen Eimer treffen}) = 0.5 = \frac{1}{2}$

$$P(\text{alle fünf Bälle in denselben Eimer}) = \left(\frac{1}{6}\right)^5 + \left(\frac{1}{3}\right)^5 + \left(\frac{1}{2}\right)^5 = \frac{23}{648} \approx 3.55\%$$